

Linear stability of RMHD equations on 2D finite channel

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Abstract

This is a note about linear stability of RMHD equations on 2D finite channel.

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1 Introduction

Consider the two dimensional magnetohydrodynamic equation on a finite channel $\Omega = [0, 1] \times \mathbb{T}$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, with small magnetic resistivity but without viscosity. We are interested in its linear stability around the equilibrium of zero velocity field and exponential shear magnetic field. The equilibrium is $V_s = 0$, $B_s = (0, b(x))$, $P_s = 0$, total magnetic field is $\tilde{B} = B_s + B$, where B is the perturbation. We assume that b is monotone positive, and currently I want to impose a restriction on it, so that $b(x) = e^{\lambda x}$ for some constant parameter $\lambda \in \mathbb{R}$. Total velocity field is $V_s + v = v$, total pressure is $P_s + p = p$, and they are the same as the perturbations v, p , then the original RMHD system has the following form:

$$\partial_t v + v \cdot \nabla v + \nabla p = \tilde{B} \cdot \nabla \tilde{B}, \quad (1)$$

$$\partial_t \tilde{B} - \eta \Delta \tilde{B} + v \cdot \nabla \tilde{B} = \tilde{B} \cdot \nabla v, \quad (2)$$

$$\nabla \cdot v = \nabla \cdot \tilde{B} = 0, \quad (3)$$

Nonlinear equations for the perturbations are

$$\begin{aligned} \partial_t v_x &= -\partial_x p - (v_x \partial_x + v_y \partial_y) v_x + (B_x \partial_x + (b + B_y) \partial_y) B_x, \\ \partial_t v_y &= -\partial_y p - (v_x \partial_x + v_y \partial_y) v_y + B_x \partial_x (b + B_y) + (b + B_y) \partial_y B_y, \\ \partial_t B_x &= (B_x \partial_x + (b + B_y) \partial_y) v_x - (v_x \partial_x + v_y \partial_y) B_x + \eta \Delta B_x, \\ \partial_t B_y &= (B_x \partial_x + (b + B_y) \partial_y) v_y - v_x \partial_x (b + B_y) - v_y \partial_y B_y + \eta \Delta B_y, \\ \nabla \cdot v &= \nabla \cdot B = 0, \end{aligned}$$

Linearized equations for the perturbations are:

$$\partial_t v_x = -\partial_x p + b\partial_y B_x, \quad (4)$$

$$\partial_t v_y = -\partial_y p + b\partial_y B_y + b' B_x, \quad (5)$$

$$\partial_t B_x = b\partial_y v_x + \eta\Delta B_x, \quad (6)$$

$$\partial_t B_y = b\partial_y v_y - b' v_x + \eta\Delta B_y, \quad (7)$$

$$\nabla \cdot v = \nabla \cdot B = 0, \quad (8)$$

with Navier slip boundary conditions

$$v_x|_{x=0,1} = B_x|_{x=0,1} = 0,$$

Taking Fourier transform in y , we get for $\alpha \neq 0$,

$$\partial_t \widehat{v}_x = -\partial_x \widehat{p} + i\alpha b \widehat{B}_x, \quad (9)$$

$$\partial_t \widehat{v}_y = -i\alpha \widehat{p} + i\alpha b \widehat{B}_y + b' \widehat{B}_x, \quad (10)$$

$$\partial_t \widehat{B}_x = i\alpha b \widehat{v}_x + \eta(\partial_x^2 - \alpha^2) \widehat{B}_x, \quad (11)$$

$$\partial_t \widehat{B}_y = i\alpha b \widehat{v}_y - b' \widehat{v}_x + \eta(\partial_x^2 - \alpha^2) \widehat{B}_y, \quad (12)$$

$$\partial_x \widehat{v}_x + i\alpha \widehat{v}_y = 0, \quad \partial_x \widehat{B}_x + i\alpha \widehat{B}_y = 0, \quad (13)$$

Eliminating $\widehat{p}, \widehat{v}_y, \widehat{B}_y$ from [equation 10](#) gives

$$\partial_t \partial_x \widehat{v}_x = \partial_t (-i\alpha \widehat{v}_y) = -\alpha^2 \widehat{p} + i\alpha b \partial_x \widehat{B}_x - i\alpha b' \widehat{B}_x,$$

$$\partial_t \partial_x^2 \widehat{v}_x = -\alpha^2 \partial_x \widehat{p} + i\alpha \partial_x (b \partial_x \widehat{B}_x) - i\alpha \partial_x (b' \widehat{B}_x), \quad \partial_x \widehat{p} = -\partial_t \widehat{v}_x + i\alpha b \widehat{B}_x,$$

$$\begin{aligned} \partial_t \partial_x^2 \widehat{v}_x &= -\alpha^2 (-\partial_t \widehat{v}_x + i\alpha b \widehat{B}_x) + i\alpha \partial_x (b \partial_x \widehat{B}_x) - i\alpha \partial_x (b' \widehat{B}_x) \\ &= -\alpha^2 (-\partial_t \widehat{v}_x + i\alpha b \widehat{B}_x) + i\alpha (b \partial_x^2 \widehat{B}_x - b'' \widehat{B}_x), \end{aligned}$$

$$\partial_t (\partial_x^2 - \alpha^2) \widehat{v}_x = \alpha b (\partial_x^2 - \alpha^2) i \widehat{B}_x - \alpha b'' i \widehat{B}_x, \quad \partial_t i \widehat{B}_x = -\alpha b \widehat{v}_x + \eta (\partial_x^2 - \alpha^2) i \widehat{B}_x,$$

Let $\xi = \widehat{v}_x, \psi = i \widehat{B}_x$, we have the following system of evolution:

$$\partial_t \xi = \alpha (\partial_x^2 - \alpha^2)^{-1} (b (\partial_x^2 - \alpha^2) \psi - b'' \psi), \quad (14)$$

$$\partial_t \psi = -\alpha b \xi + \eta (\partial_x^2 - \alpha^2) \psi, \quad (15)$$

Denote \mathcal{L}_α the linear operator of the above equations, and consider the eigenvalue problem of operator \mathcal{L}_α . If $c \in \sigma_p(\mathcal{L}_\alpha)$ with associated eigenfunctions ξ, ψ , then

$$c\xi = \alpha (\partial_x^2 - \alpha^2)^{-1} (b (\partial_x^2 - \alpha^2) \psi - b'' \psi), \quad c\psi = -\alpha b \xi + \eta (\partial_x^2 - \alpha^2) \psi,$$

$$\alpha b \xi = -(c - \eta (\partial_x^2 - \alpha^2)) \psi, \quad c (\partial_x^2 - \alpha^2) \xi = \alpha (b (\partial_x^2 - \alpha^2) - b'') \psi,$$

$$-c (\partial_x^2 - \alpha^2) b^{-1} (c - \eta (\partial_x^2 - \alpha^2)) \psi = \alpha^2 (b (\partial_x^2 - \alpha^2) - b'') \psi,$$

Let $\psi = bg$, then

$$b^{-1} (c - \eta (\partial_x^2 - \alpha^2)) bg = cg - c\eta b^{-1} (\partial_x^2 - \alpha^2) bg,$$

$$\alpha^2 (b (\partial_x^2 - \alpha^2) - b'') bg = \alpha^2 (b^2 (\partial_x^2 - \alpha^2) + 2bb' \partial_x) g,$$

Summing up the above two equations, we get the Orr-Sommerfeld type equation for linearized RMHD system on a 2-dimensional finite channel:

$$(c^2 + \alpha^2 b^2)(\partial_x^2 - \alpha^2)g + 2\alpha^2 b b' \partial_x g - c\eta(\partial_x^2 - \alpha^2)b^{-1}(\partial_x^2 - \alpha^2)bg = 0, \quad (16)$$

Boundary conditions are

$$\begin{aligned} \psi|_{x=0,1} = 0, \quad \xi|_{x=0,1} = 0, \quad (c - \eta(\partial_x^2 - \alpha^2))\psi|_{x=0,1} = 0, \\ g|_{x=0,1} = 0, \quad \partial_x^2(bg)|_{x=0,1} = b\partial_x^2 g + 2b'\partial_x g|_{x=0,1} = 0, \end{aligned}$$

Let us denote by OS_α the Orr-Sommerfeld type fourth-order operator

$$OS_\alpha(g) \triangleq (c^2 + \alpha^2 b^2)(\partial_x^2 - \alpha^2)g + 2\alpha^2 b b' \partial_x g - c\eta(\partial_x^2 - \alpha^2)b^{-1}(\partial_x^2 - \alpha^2)bg,$$

Following [11], we study the resolvent estimates of the linearized operator under the Navier-slip boundary conditions. More precisely, we consider the equation

$$OS_\alpha(g) = (c^2 + \alpha^2 b^2)(\partial_x^2 - \alpha^2)g + 2\alpha^2 b b' \partial_x g - c\eta(\partial_x^2 - \alpha^2)b^{-1}(\partial_x^2 - \alpha^2)bg = F,$$

$$g|_{x=0,1} = 0, \quad b\partial_x^2 g + 2b'\partial_x g|_{x=0,1} = 0,$$

Substitute $h = b^{-1}(\partial_x^2 - \alpha^2)bg = b^{-1}(\partial_x^2 - \alpha^2)\psi$, we have

$$h|_{x=0,1} = 0, \quad g = b^{-1}(\partial_x^2 - \alpha^2)^{-1}bh,$$

$$OS_\alpha(g) = (c^2 + \alpha^2 b^2)(\partial_x^2 - \alpha^2)b^{-1}(\partial_x^2 - \alpha^2)^{-1}bh + 2\alpha^2 b b' \partial_x b^{-1}(\partial_x^2 - \alpha^2)^{-1}bh - c\eta(\partial_x^2 - \alpha^2)h,$$

Using Green's function, the operator $(\partial_x^2 - \alpha^2)^{-1}$ with Dirichlet's boundary condition can be represented as an integral operator:

$$G(x, x') = \begin{cases} -\frac{\sinh \alpha(1-x') \sinh \alpha x}{\alpha \sinh \alpha}, & x \leq x' \\ -\frac{\sinh \alpha x' \sinh \alpha(1-x)}{\alpha \sinh \alpha}, & x > x' \end{cases}, \quad (\partial_x^2 - \alpha^2)^{-1}u(x) = \int_0^1 G(x, x')u(x')dx',$$

$$g(x) = b^{-1}(x)[(\partial_x^2 - \alpha^2)^{-1}bh](x) = b^{-1}(x) \int_0^1 G(x, x')b(x')h(x')dx',$$

As an integral operator, $\partial_x(\partial_x^2 - \alpha^2)^{-1}$ is represented as follows:

$$\partial_x G(x, x') = \begin{cases} -\frac{\alpha \sinh \alpha(1-x') \cosh \alpha x}{\alpha \sinh \alpha}, & x \leq x' \\ \frac{\alpha \sinh \alpha x' \cosh \alpha(1-x)}{\alpha \sinh \alpha}, & x > x' \end{cases}, \quad \partial_x(\partial_x^2 - \alpha^2)^{-1}u(x) = \int_0^1 \partial_x G(x, x')u(x')dx',$$

We investigate the case of exponential background magnetic profile $b(x) = e^{\lambda x}$ for convenience. The Sobolev space we concern is

$$H_0^1([0, 1]) = \{u : [0, 1] \rightarrow \mathbb{C}, \|u\|_{H^1} < +\infty, u(0) = u(1) = 0\}, \quad \|u\|_{H^1}^2 = \int_0^1 \partial_x u \overline{\partial_x u} dx,$$

A set of orthonormal basis of $H_0^1([0, 1])$ is $\{e_k = \frac{\sqrt{2} \sin k\pi x}{k\pi}, k \in \mathbb{Z}_+\}$, and they are all the eigenfunctions of operator ∂_x^2 , with eigenvalues $\partial_x^2 e_k = -k^2 \pi^2 e_k$. Under the exponential background magnetic profile, we have

$$(\partial_x^2 - \alpha^2)g = (\partial_x^2 - \alpha^2)b^{-1}(\partial_x^2 - \alpha^2)^{-1}bh = h + 2(b^{-1})'\partial_x(\partial_x^2 - \alpha^2)^{-1}bh + (b^{-1})''(\partial_x^2 - \alpha^2)^{-1}bh,$$

$$\partial_x g = \partial_x b^{-1}(\partial_x^2 - \alpha^2)^{-1}bh = (b^{-1})'(\partial_x^2 - \alpha^2)^{-1}bh + b^{-1}\partial_x(\partial_x^2 - \alpha^2)^{-1}bh,$$

The integral form of the above two equations are:

$$\begin{aligned}(\partial_x^2 - \alpha^2)g(x) &= h(x) + \lambda b^{-1}(x)[(\lambda - 2\partial_x)(\partial_x^2 - \alpha^2)^{-1}bh](x) \\ &= h(x) + \lambda b^{-1}(x) \int_0^1 (\lambda - 2\partial_x)G(x, x')b(x')h(x')dx',\end{aligned}$$

$$\partial_x g(x) = b^{-1}(x)[(\partial_x - \lambda)(\partial_x^2 - \alpha^2)^{-1}bh](x) = b^{-1}(x) \int_0^1 (\partial_x - \lambda)G(x, x')b(x')h(x')dx',$$

The Orr-Sommerfeld type equation in terms of h becomes

$$\begin{aligned}F \triangleq OS_\alpha(g) &= (c^2 + \alpha^2 b^2)(h + 2(b^{-1})'\partial_x(\partial_x^2 - \alpha^2)^{-1}bh + (b^{-1})''(\partial_x^2 - \alpha^2)^{-1}bh) \\ &\quad + 2\alpha^2 b b'((b^{-1})'(\partial_x^2 - \alpha^2)^{-1}bh + b^{-1}\partial_x(\partial_x^2 - \alpha^2)^{-1}bh) - c\eta(\partial_x^2 - \alpha^2)h,\end{aligned}$$

$$\begin{aligned}F &= (c^2 + \alpha^2 b^2)(h - 2\lambda b^{-1}\partial_x(\partial_x^2 - \alpha^2)^{-1}bh + \lambda^2 b^{-1}(\partial_x^2 - \alpha^2)^{-1}bh) + 2\alpha^2 \lambda b(\partial_x - \lambda)(\partial_x^2 - \alpha^2)^{-1}bh - c\eta(\partial_x^2 - \alpha^2)h, \\ F &= c^2(h - 2\lambda b^{-1}\partial_x(\partial_x^2 - \alpha^2)^{-1}bh + \lambda^2 b^{-1}(\partial_x^2 - \alpha^2)^{-1}bh) + \alpha^2 b^2 h - \lambda^2 \alpha^2 b(\partial_x^2 - \alpha^2)^{-1}bh - c\eta(\partial_x^2 - \alpha^2)h \\ &= c^2(h + \lambda b^{-1}(\lambda - 2\partial_x)(\partial_x^2 - \alpha^2)^{-1}bh) + \alpha^2 b^2 h - \lambda^2 \alpha^2 b(\partial_x^2 - \alpha^2)^{-1}bh - c\eta(\partial_x^2 - \alpha^2)h,\end{aligned}$$

While taking inner product with h , the first and the third term in the right hand side above is nontrivial. We give a closer look at the first term below:

$$(\partial_x^2 - \alpha^2)g = h + \lambda b^{-1}(\lambda - 2\partial_x)(\partial_x^2 - \alpha^2)^{-1}bh,$$

$$\begin{aligned}\langle (\partial_x^2 - \alpha^2)g, h \rangle &= \int_0^1 b^{-1}(\partial_x^2 - \alpha^2)g(\partial_x^2 - \alpha^2)b\bar{g} = \int_0^1 b^{-1}(\partial_x^2 - \alpha^2)g(b(\partial_x^2 - \alpha^2)\bar{g} + \lambda^2 b\bar{g} + 2\lambda b\partial_x \bar{g}) \\ &= \|(\partial_x^2 - \alpha^2)g\|_2^2 + \lambda^2 \int_0^1 (\partial_x^2 - \alpha^2)g\bar{g} + 2\lambda \int_0^1 (\partial_x^2 - \alpha^2)g \cdot \partial_x \bar{g} \\ &= \|(\partial_x^2 - \alpha^2)g\|_2^2 - \lambda^2 \|g'\|_2^2 - \alpha^2 \lambda^2 \|g\|_2^2 + 2\lambda(\langle g'', g' \rangle - \alpha^2 \langle g, g' \rangle), \\ &\quad \|(\partial_x^2 - \alpha^2)g\|_2^2 = \|g''\|_2^2 + \alpha^4 \|g\|_2^2 + 2\alpha^2 \|g'\|_2^2,\end{aligned}$$

The third term is dealt with as follows:

$$b(\partial_x^2 - \alpha^2)^{-1}bh = b\psi, \quad h = b^{-1}(\partial_x^2 - \alpha^2)\psi,$$

$$\langle b\psi, h \rangle = \int_0^1 \psi(\partial_x^2 - \alpha^2)\bar{\psi}dx = -\|\psi'\|_2^2 - \alpha^2 \|\psi\|_2^2,$$

Combining the equations above together, we have

$$\begin{aligned}\langle F, ch \rangle &= \langle c^2(\partial_x^2 - \alpha^2)g, ch \rangle + \bar{c}\alpha^2 \|bh\|_2^2 - \bar{c}\lambda^2 \alpha^2 \langle b\psi, h \rangle - |c|^2 \eta \langle (\partial_x^2 - \alpha^2)h, h \rangle \\ &= |c|^2 c (\|(\partial_x^2 - \alpha^2)g\|_2^2 - \lambda^2 \|g'\|_2^2 - \alpha^2 \lambda^2 \|g\|_2^2 + 2\lambda(\langle g'', g' \rangle - \alpha^2 \langle g, g' \rangle)) \\ &\quad + \bar{c}(\alpha^2 \|bh\|_2^2 + \lambda^2 \alpha^2 \|\psi'\|_2^2 + \lambda^2 \alpha^4 \|\psi\|_2^2) + |c|^2 \eta (\|h'\|_2^2 + \alpha^2 \|h\|_2^2) \\ &= |c|^2 c (\|g''\|_2^2 + (2\alpha^2 - \lambda^2)\|g'\|_2^2 + \alpha^2(\alpha^2 - \lambda^2)\|g\|_2^2 + 2\lambda(\langle g'', g' \rangle - \alpha^2 \langle g, g' \rangle)) \\ &\quad + \bar{c}(\alpha^2 \|bh\|_2^2 + \lambda^2 \alpha^2 \|\psi'\|_2^2 + \lambda^2 \alpha^4 \|\psi\|_2^2) + |c|^2 \eta (\|h'\|_2^2 + \alpha^2 \|h\|_2^2)\end{aligned}$$

We see that when $\Re(c) > 0$, $-1 \leq \lambda \leq 1$, the real part of the right hand side of the above equation is strictly positive for non-zero h .

$$\|bh\|_2^2 = \|(\partial_x^2 - \alpha^2)\psi\|_2^2 = \|\psi''\|_2^2 + 2\alpha^2 \|\psi'\|_2^2 + \alpha^4 \|\psi\|_2^2,$$

Question 1. 1) Prove that when $\eta = 0$, if $c \in \sigma(\mathcal{L}_\alpha)$ then there exist $x_c \in [0, 1]$ such that $c = \pm i\alpha b(x_c)$. It means that \mathcal{L}_α can only have embedding eigenvalues. An equivalent form of this proposition appears in [10].

2) Does the Rayleigh equation for Euler's equation only admits embedding eigenvalues?

Proof. Method 1: When $c^2 + \alpha^2 b^2 \neq 0$, rewrite the equation as follows

$$(\partial_x^2 - \alpha^2)g + \frac{2\alpha^2 b b' \partial_x g}{c^2 + \alpha^2 b^2} = 0, \quad g(0) = g(1) = 0,$$

It is an second order elliptic ordinary differential equation on $[0, 1]$ with Dirichlet's boundary condition. □

$$\begin{aligned} \mathcal{H} &= \{(\psi, \xi), \psi, \xi \in H_0^1([0, 1])\}, \quad \|(\psi, \xi)\|_{\mathcal{H}}^2 = \int_0^1 \partial_x \psi \overline{\partial_x \psi} + \partial_x \xi \overline{\partial_x \xi}, \\ \partial_t \xi &= \alpha(\partial_x^2 - \alpha^2)^{-1}(b(\partial_x^2 - \alpha^2)\psi - b''\psi) = \alpha(b\psi + K_1\psi), \end{aligned}$$

where K_1 is a compact operator defined by

$$K_1\psi = (\partial_x^2 - \alpha^2)^{-1}(-2b'\partial_x\psi - 2b''\psi) = -2(\partial_x^2 - \alpha^2)^{-1}\partial_x(b'\psi),$$

and we have $\overline{K_1\psi} = K_1\overline{\psi}$. Notice that $v_x, B_x \in \mathbb{R}, \xi = \widehat{v}_x, \psi = i\widehat{B}_x$ only implies that $\widehat{v}_x(\alpha) = \overline{\widehat{v}_x(-\alpha)}, \widehat{B}_x(\alpha) = \overline{\widehat{B}_x(-\alpha)}$, ξ, ψ are complex-valued functions.

$$\int_0^1 \partial_t \xi \overline{\xi} = \int_0^1 \alpha(b\psi + K_1\psi)\overline{\xi}, \quad \int_0^1 \partial_t \psi \overline{\psi} = \int_0^1 (-\alpha b\xi + \eta(\partial_x^2 - \alpha^2)\psi)\overline{\psi},$$

$$\partial_t(\|\psi\|_2^2 + \|\xi\|_2^2) = \int_0^1 \alpha(K_1\psi\overline{\xi} + \xi K_1\overline{\psi}) + \eta(\partial_x^2 - \alpha^2)\psi\overline{\psi} + \psi\eta(\partial_x^2 - \alpha^2)\overline{\psi},$$

When $b(x) = e^{\lambda x}$, we have $K_1\psi = (\partial_x^2 - \alpha^2)^{-1}(-2\lambda b\partial_x\psi - 2\lambda^2 b\psi) = -2\lambda(\partial_x^2 - \alpha^2)^{-1}\partial_x(b\psi)$.

$$\partial_t \psi = -\alpha b\xi + \eta(\partial_x^2 - \alpha^2)\psi = -\alpha b(\xi(0) + \int_0^t \alpha(b\psi(t') + K\psi(t'))dt') + \eta(\partial_x^2 - \alpha^2)\psi,$$

So the evolutionary equation of ψ takes the following form, which is similar to the wave equation:

$$\partial_t^2 \psi = -\alpha^2 b(b\psi + K_1\psi) + \eta(\partial_x^2 - \alpha^2)\partial_t \psi,$$

Substitute $\psi = bg$ and let $\eta = 0, B = b^2$, we have

$$\partial_t^2(bg) = -\alpha^2 b(b^2g + Kbg), \quad K_1bg = -2(\partial_x^2 - \alpha^2)^{-1}\partial_x(b'bg) = -(\partial_x^2 - \alpha^2)^{-1}\partial_x(B'g),$$

$$\partial_t^2 g + \alpha^2(b^2g + K_1bg) = \partial_t^2 g + \alpha^2(Bg - (\partial_x^2 - \alpha^2)^{-1}\partial_x(B'g)) = 0,$$

the above calculation is the same as equation (2.5) in [10]. We define

$$Kg = K_1bg = -(\partial_x^2 - \alpha^2)^{-1}\partial_x(B'g),$$

Proposition 1 (Energy conservation on each frequency).

2 Spectral method

Theorem 1. Reference: Lecture notes on functional analysis II by Gongqing Zhang, p53 problem 5.5.10. \mathcal{H} is a Hilbert space, N is a normal operator on \mathcal{H} and its spectrum $\sigma(N)$ is countable, then \mathcal{H} has a orthonormal basis $B = \{y\}$ where y are eigenfunctions of N , and the Fourier expansion holds:

$$x = \sum_{y \in B} (x, y)y, \quad x \in \mathcal{H},$$

the Fourier coefficients (x, y) only have countably many nonzero elements.

Proof. 1) Eigenspaces of different eigenvalues are orthogonal. If f_1, f_2 are two eigenfunctions of N with different eigenvalues λ_1, λ_2 . When N is self-adjoint, we have $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\lambda_1 \langle f_1, f_2 \rangle = \langle Nf_1, f_2 \rangle = \langle f_1, Nf_2 \rangle = \lambda_2 \langle f_1, f_2 \rangle,$$

Since $\lambda_1 \neq \lambda_2$, $\langle f_1, f_2 \rangle = 0$. □

Problem 1. Eigenfunctions of ∂_x^2 on $H_0^1([0, 1])$.

$$\partial_x^2 f = \lambda f, \quad -\xi^2 \widehat{f} = \lambda \widehat{f}, \quad \lambda = -\xi^2, \quad \text{supp } \widehat{f} = \{\pm \xi\}, \quad f(x) = \widehat{f}(\xi)e^{i\xi x} + \widehat{f}(-\xi)e^{-i\xi x},$$

Boundary conditions are

$$f(0) = \widehat{f}(\xi) + \widehat{f}(-\xi) = 0, \quad f(1) = \widehat{f}(\xi)e^{i\xi} + \widehat{f}(-\xi)e^{-i\xi} = 0,$$

So we have $e^{i\xi} = e^{-i\xi}$, $\xi = k\pi$, $k \in \mathbb{Z} \setminus \{0\}$, $\lambda = -k^2\pi^2$, $f = C \sin k\pi x$.

Consider the case when $b(x)$ is a positive constant. We have

$$OS_\alpha(g) = (c^2 + \alpha^2 b^2)(\partial_x^2 - \alpha^2)g - c\eta(\partial_x^2 - \alpha^2)^2 g, \quad g|_{x=0,1} = g'|_{x=0,1} = 0,$$

Let $h = (\partial_x^2 - \alpha^2)g$, we claim that when $\Re(c) > 0$, $f \in H_0^1([0, 1])$, the solution to $OS_\alpha(g) = f$ uniquely exists. We have

$$\mathcal{L}_\alpha h \triangleq OS_\alpha(g) = (c^2 + \alpha^2 b^2)h - c\eta(\partial_x^2 - \alpha^2)h = f,$$

Assume that $f = \sum_k (f, e_k)e_k$, then $(\partial_x^2 - \alpha^2)e_k = -(k^2\pi^2 + \alpha^2)e_k$,

$$(f, e_k) = (h, e_k)(c^2 + \alpha^2 b^2 + c\eta(k^2\pi^2 + \alpha^2)), \quad (h, e_k) = \frac{(f, e_k)}{c^2 + \alpha^2 b^2 + c\eta(k^2\pi^2 + \alpha^2)},$$

$$(g, e_k) = -\frac{(h, e_k)}{(k^2\pi^2 + \alpha^2)} = -\frac{(f, e_k)}{(c^2 + \alpha^2 b^2 + c\eta(k^2\pi^2 + \alpha^2))(k^2\pi^2 + \alpha^2)},$$

Problem 2. Eigenvalue problem: when b is constant, for what values of c is \mathcal{L}_α not injective?

$$\mathcal{L}_\alpha h = (c^2 + \alpha^2 b^2)h - c\eta(\partial_x^2 - \alpha^2)h = 0,$$

When $\eta = 0$, eigenvalues are $c = \pm i\alpha b$, and their invariant space is the whole $H_0^1([0, 1])$. When $\eta > 0$, 1) $c^2 + \alpha^2 b^2 = 0$, then $c \neq 0$, since $\partial_x^2 - \alpha^2$ is injective, we have $h = 0$.

2) $c^2 + \alpha^2 b^2 \neq 0$, take Fourier transform $h = \sum_k \widehat{h}_k e_k$, we have

$$(c^2 + \alpha^2 b^2 + c\eta\alpha^2 + c\eta k^2\pi^2)\widehat{h}_k = 0, \quad k \neq 0,$$

For small k , $\eta\alpha^2 + \eta k^2\pi^2 \leq 2|\alpha|b$, the roots are two conjugate complex numbers with negative real parts and norm αb . For large k , $\eta\alpha^2 + \eta k^2\pi^2 > 2|\alpha|b$, the two real roots are

$$c_{1,2} = -\frac{\eta\alpha^2 + \eta k^2\pi^2 \pm \sqrt{(\eta\alpha^2 + \eta k^2\pi^2)^2 - 4\alpha^2 b^2}}{2},$$

$$A = \int_0^1 e^{\lambda x} \sin k\pi x dx = - \int_0^1 \frac{e^{\lambda x}}{\lambda} k\pi \cos k\pi x dx = \frac{k\pi}{\lambda^2} - (-1)^k e^{\lambda} \frac{k\pi}{\lambda^2} - \int_0^1 \frac{e^{\lambda x}}{\lambda^2} k^2 \pi^2 \sin k\pi x dx,$$

$$\int_0^1 \frac{e^{\lambda x}}{\lambda^2} k^2 \pi^2 \sin k\pi x dx = \frac{k^2 \pi^2}{\lambda^2} A, \quad A = \frac{(1 - (-1)^k e^{\lambda}) k\pi}{\lambda^2 + k^2 \pi^2},$$

$$B = \int_0^1 e^{\lambda x} \cos m\pi x dx = (-1)^m \frac{e^{\lambda}}{\lambda} - \frac{1}{\lambda} + \int_0^1 \frac{e^{\lambda x}}{\lambda} m\pi \sin m\pi x dx,$$

$$\int_0^1 \frac{e^{\lambda x}}{\lambda} m\pi \sin m\pi x dx = - \int_0^1 \frac{e^{\lambda x}}{\lambda^2} m^2 \pi^2 \sin m\pi x dx = - \frac{m^2 \pi^2}{\lambda^2} B, \quad B = \frac{((-1)^m e^{\lambda} - 1)\lambda}{\lambda^2 + m^2 \pi^2},$$

When $\lambda < 0$, we may also use complex analysis techniques to calculate the above integrals:

$$\int_0^{+\infty} e^{\lambda x} \sin k\pi x dx = \Im \int_0^{+\infty} e^{(\lambda + ik\pi)x} dx = \Im \frac{1}{-\lambda - ik\pi} = \frac{k\pi}{\lambda^2 + k^2 \pi^2},$$

$$\int_0^{+\infty} e^{\lambda x} \sin k\pi x dx = A(1 + (-1)^k e^{\lambda} + (-1)^{2k} e^{2\lambda} + \dots) = \frac{A}{1 - (-1)^k e^{\lambda}}, \quad A = \frac{(1 - (-1)^k e^{\lambda}) k\pi}{\lambda^2 + k^2 \pi^2},$$

$$\int_0^{+\infty} e^{\lambda x} \cos m\pi x dx = \Re \int_0^{+\infty} e^{(\lambda + im\pi)x} dx = \Re \frac{1}{-\lambda - im\pi} = \frac{-\lambda}{\lambda^2 + m^2 \pi^2},$$

$$\int_0^{+\infty} e^{\lambda x} \cos m\pi x dx = B(1 + (-1)^m e^{\lambda} + (-1)^{2m} e^{2\lambda} + \dots) = \frac{B}{1 - (-1)^m e^{\lambda}}, \quad B = \frac{((-1)^m e^{\lambda} - 1)\lambda}{\lambda^2 + m^2 \pi^2},$$

I try to express the relationship of h and $g = b^{-1}(\partial_x^2 - \alpha^2)^{-1}bh$ on the frequency domain. Recall that $e_k = \frac{\sqrt{2} \sin k\pi x}{k\pi}$, $k \in \mathbb{Z}_+$,

$$g = \sum_k \widehat{g}_k e_k, \quad \widehat{g}_k = \langle g, e_k \rangle = \int_0^1 \partial_x g \overline{\partial_x e_k} dx,$$

$$h = \sum_k \widehat{h}_k e_k, \quad \widehat{h}_k = \langle h, e_k \rangle = \int_0^1 \partial_x h \overline{\partial_x e_k} dx,$$

$$(\widehat{bg})_k = \langle bg, e_k \rangle = \int_0^1 \partial_x (bg) \overline{\partial_x e_k} = - \int_0^1 bg \overline{\partial_x^2 e_k} = \int_0^1 bg k^2 \pi^2 \overline{e_k}$$

$$= \int_0^1 b \left(\sum_l \widehat{g}_l e_l \right) k^2 \pi^2 \overline{e_k} = \sum_l \widehat{g}_l \int_0^1 b k^2 \pi^2 e_l \overline{e_k},$$

$$e_l \overline{e_k} = \frac{2 \sin l\pi x \sin k\pi x}{lk\pi^2} = \frac{\cos(l-k)\pi x - \cos(l+k)\pi x}{lk\pi^2},$$

$$\int_0^1 b k^2 \pi^2 e_l \overline{e_k} = \int_0^1 b \frac{k}{l} (\cos(l-k)\pi x - \cos(l+k)\pi x),$$

$$\int_0^1 b \frac{k}{l} \cos(l-k)\pi x = \frac{k}{l} \int_0^1 e^{\lambda x} \cos(l-k)\pi x dx = \frac{k}{l} \frac{((-1)^{l-k} e^{\lambda} - 1)\lambda}{\lambda^2 + (l-k)^2 \pi^2} = B_1,$$

$$\int_0^1 b \frac{k}{l} \cos(l+k)\pi x = \frac{k}{l} \frac{((-1)^{l+k} e^{\lambda} - 1)\lambda}{\lambda^2 + (l+k)^2 \pi^2} = B_2,$$

$$B_1 - B_2 = \frac{k}{l}((-1)^{l-k}e^\lambda - 1)\lambda\left(\frac{1}{\lambda^2 + (l-k)^2\pi^2} - \frac{1}{\lambda^2 + (l+k)^2\pi^2}\right) = \frac{4k^2\pi^2((-1)^{l-k}e^\lambda - 1)\lambda}{(\lambda^2 + (l^2 + k^2)\pi^2)^2 - 4l^2k^2\pi^4},$$

$$(\widehat{bg})_k = \sum_l \widehat{g}_l \frac{4k^2\pi^2((-1)^{l-k}e^\lambda - 1)\lambda}{(\lambda^2 + (l^2 + k^2)\pi^2)^2 - 4l^2k^2\pi^4},$$

If $u, v \in H_0^1([0, 1])$, $u = \sum_k \widehat{u}_k e_k$, $v = \sum_k \widehat{v}_k e_k$ satisfy $v = (\partial_x^2 - \alpha^2)u$, then their Fourier coefficients satisfy

$$\widehat{v}_k = -(k^2\pi^2 + \alpha^2)\widehat{u}_k, \quad \widehat{u}_k = -\frac{\widehat{v}_k}{k^2\pi^2 + \alpha^2},$$

Since $b^{-1}(x) = e^{-\lambda x}$, the action of multiplier b^{-1} on the Fourier side is given by

$$(\widehat{b^{-1}g})_k = \sum_l \widehat{g}_l \frac{4k^2\pi^2((-1)^{l-k}e^{-\lambda} - 1)(-\lambda)}{(\lambda^2 + (l^2 + k^2)\pi^2)^2 - 4l^2k^2\pi^4},$$

We may denote

$$K_\lambda(k, l) = \frac{4k^2\pi^2}{(\lambda^2 + (l^2 + k^2)\pi^2)^2 - 4l^2k^2\pi^4},$$

Consider representing the Fourier coefficients of h in terms of Fourier coefficients of g , where $g = b^{-1}(\partial_x^2 - \alpha^2)^{-1}bh$, we have

$$v = bh, \quad \psi = (\partial_x^2 - \alpha^2)^{-1}v, \quad g = b^{-1}\psi,$$

$$\widehat{v}_k = \sum_l \widehat{h}_l((-1)^{l-k}e^\lambda - 1)\lambda K_\lambda(k, l), \quad \widehat{\psi}_k = -\frac{\widehat{v}_k}{k^2\pi^2 + \alpha^2},$$

$$\begin{aligned} \widehat{g}_m &= \sum_k \widehat{\psi}_k((-1)^{l-k}e^{-\lambda} - 1)(-\lambda)K_\lambda(m, k) = \sum_k \widehat{v}_k \frac{((-1)^{l-k}e^{-\lambda} - 1)\lambda}{k^2\pi^2 + \alpha^2} K_\lambda(m, k) \\ &= \sum_{k,l} \frac{((-1)^{k-m}e^{-\lambda} - 1)\lambda}{k^2\pi^2 + \alpha^2} K_\lambda(m, k)((-1)^{l-k}e^\lambda - 1)\lambda K_\lambda(k, l)\widehat{h}_l \\ &= \sum_l \widehat{h}_l \sum_k \frac{\lambda^2((-1)^{k-m}e^{-\lambda} - 1)((-1)^{l-k}e^\lambda - 1)}{k^2\pi^2 + \alpha^2} K_\lambda(m, k)K_\lambda(k, l) \end{aligned}$$

3 Energy conservation of nonlinear RVMHD

Suppose u and b are the velocity field and magnetic field, p is the total pressure, we consider the original nonlinear RVMHD equations' energy conservation:

$$\partial_t(\|u\|_2^2 + \|b\|_2^2) = \partial_t \int_\Omega u \cdot u + b \cdot b = 2 \int_\Omega u \cdot (-\nabla p - u \cdot \nabla u + b \cdot \nabla b + \mu \Delta u) + b \cdot (b \cdot \nabla u - u \cdot \nabla b + \eta \Delta b),$$

$$\int_\Omega u \cdot \nabla p = \int_\Omega u \cdot \nabla p + p \nabla \cdot u = \int_\Omega \nabla \cdot (pu) = 0,$$

$$u \cdot (u \cdot \nabla u) = u \cdot \nabla \frac{|u|^2}{2}, \quad \nabla \cdot (|u|^2 u) = u \cdot \nabla |u|^2 + |u|^2 \nabla \cdot u,$$

$$\nabla \cdot (|b|^2 u) = u \cdot \nabla |b|^2 + |b|^2 \nabla \cdot u, \quad b \cdot (u \cdot \nabla) b = u \cdot \nabla \frac{|b|^2}{2},$$

$$\begin{aligned}
\int_{\Omega} u \cdot (u \cdot \nabla u) &= \frac{1}{2} \int_{\Omega} \nabla \cdot (|u|^2 u) = 0, & \int_{\Omega} b \cdot (u \cdot \nabla b) &= \frac{1}{2} \int_{\Omega} \nabla \cdot (|b|^2 u) = 0, \\
\nabla \cdot ((u \cdot b)b) &= (b \cdot \nabla)(u \cdot b) + (u \cdot b)(\nabla \cdot b), & (b \cdot \nabla)(u \cdot b) &= u \cdot (b \cdot \nabla)b + b \cdot (b \cdot \nabla)u, \\
\int_{\Omega} u \cdot (b \cdot \nabla)b + b \cdot (b \cdot \nabla)u &= \int_{\Omega} \nabla \cdot ((u \cdot b)b) = 0, \\
\int_{\Omega} u \cdot \Delta u &= - \int_{\Omega} \nabla u_x \cdot \nabla u_x + \nabla u_y \cdot \nabla u_y + \nabla u_z \cdot \nabla u_z = - \int_{\Omega} |\nabla u|^2, & \int_{\Omega} b \cdot \Delta b &= - \int_{\Omega} |\nabla b|^2,
\end{aligned}$$

Energy conservation for perturbations are

Representing $(\partial_x^2)^{-1}$ with Dirichlet's boundary condition on $[0, 1]$ as an integral operator:

$$\partial_x^2 v = u, \quad u|_{x=0,1} = 0, \quad v|_{x=0,1} = 0, \quad v(x) = \int_0^1 G(x, x') u(x') dx',$$

$$G(0, x') = G(1, x') = 0, \quad G(x, x') = \begin{cases} (x' - 1)x, & x \leq x' \\ (x - 1)x', & x > x' \end{cases},$$

In this case, $\partial_x^2 G(x, x') = 0$ on $[0, x')$ and $(x', 1]$, so it is linear on the above two intervals. Let

$$\partial_x G(x, x')|_{x=x'_-} = a, \quad \partial_x G(x, x')|_{x=x'_+} = b, \quad a = b - 1, \quad G(x', x') = ax' = -b(1 - x'),$$

so its solution is $a = x' - 1, b = x'$.

Representing $(\partial_x^2 - \alpha^2)^{-1}$ on $[0, 1]$ as an integral operator when $\alpha \neq 0$:

$$(\partial_x^2 - \alpha^2)v = u, \quad u|_{x=0,1} = 0, \quad v|_{x=0,1} = 0, \quad v(x) = \int_0^1 G(x, x') u(x') dx',$$

$$G(0, x') = G(1, x') = 0, \quad (\partial_x^2 - \alpha^2)v(x) = \int_0^1 (\partial_x^2 - \alpha^2)G(x, x') u(x') dx' = u(x),$$

$$g(x) = G(x, x'), \quad (\partial_x^2 - \alpha^2)g(x) = \delta_{x'}(x) = \delta(x - x'), \quad g(x) = \sum_{k \geq 1} \hat{g}(k) \sin k\pi x,$$

$$\hat{g}(k) = 2 \int_0^1 g(x) \sin k\pi x dx = 2 \int_0^1 \hat{g}(k) \sin^2 k\pi x dx = \hat{g}(k),$$

$$\hat{\delta}_{x'}(k) = 2 \int_0^1 \delta_{x'}(x) \sin k\pi x dx = 2 \sin k\pi x', \quad \hat{g}(k) = \frac{\hat{\delta}_{x'}(k)}{-k^2 \pi^2 - \alpha^2} = \frac{2 \sin k\pi x'}{-k^2 \pi^2 - \alpha^2},$$

In this case $(\partial_x^2 - \alpha^2)g(x) = 0$ on $[0, x')$ and $(x', 1]$, so it has the following forms on the above two intervals.

$$g(x) = A \sinh \alpha x, \quad x \in [0, x'), \quad g(x) = B \sinh \alpha(1 - x), \quad x \in (x', 1],$$

$$A \sinh \alpha x' = B \sinh \alpha(1 - x'), \quad \alpha A \cosh \alpha x' = -\alpha B \cosh \alpha(1 - x') - 1,$$

There exists $\lambda < 0$ such that $A = \lambda \sinh \alpha(1 - x'), B = \lambda \sinh \alpha x'$,

$$-1 = \lambda \alpha (\cosh \alpha x' \sinh \alpha(1 - x') + \cosh \alpha(1 - x') \sinh \alpha x') = \lambda \alpha \sinh \alpha, \quad \lambda = -\frac{1}{\alpha \sinh \alpha},$$

$$A = -\frac{\sinh \alpha(1 - x')}{\alpha \sinh \alpha}, \quad B = -\frac{\sinh \alpha x'}{\alpha \sinh \alpha}, \quad G(x, x') = \begin{cases} -\frac{\sinh \alpha(1-x') \sinh \alpha x}{\alpha \sinh \alpha}, & x \leq x' \\ -\frac{\sinh \alpha x' \sinh \alpha(1-x)}{\alpha \sinh \alpha}, & x > x' \end{cases},$$

Notice that if we let $\alpha \rightarrow 0_+$, the above Green's function tends to the case $\alpha = 0$.

Question 2. Δ^{-1} is the inverse of Laplacian with Dirichlet's boundary condition,

$$\Delta = \partial_r^2 - \frac{k^2}{r^2}, \quad H(r) = e^{ikB\frac{t}{r^2}}, \quad g = H^{-1}\Delta^{-1}H\Delta\psi, \quad r \in [1, 2],$$

Show that for some positive C independent to k, B, t , we have

$$\|\psi'\|_2^2 + (k^2 - \frac{1}{4})\|\frac{\psi}{r}\|_2^2 \geq C\|g'\|_2^2,$$

Modification: 1) Under Dirichlet's boundary condition,

$$\Delta = \partial_x^2 - k^2, \quad H(x) = e^{\lambda x}, \quad g = H^{-1}\Delta^{-1}H\Delta\psi, \quad x \in [0, 1],$$

Does the following inequality hold for some positive C independent to k, B, t ?

$$\|\psi'\|_2^2 + k^2\|\psi\|_2^2 \geq C\|g'\|_2^2,$$

2) Let $H(x) = e^{i\lambda x}$ be purely oscillatory, $\lambda \in \mathbb{R}_+, x \in [0, 1], g(x) \in \mathbb{R}$, then we have

$$\|\psi'\|_2^2 + k^2\|\psi\|_2^2 \geq \|g'\|_2^2 + k^2\|g\|_2^2,$$

3) Remove the condition $g(x) \in \mathbb{R}$, what can we say about ψ and g ?

Proof. 1) Let the inner product on $[0, 1]$ be $\langle u, v \rangle = \int_0^1 u\bar{v}dx$,

$$\begin{aligned} -\langle \Delta g, \psi \rangle &= -\langle g, H^{-1}\Delta Hg \rangle = -\langle H^{-1}g, \Delta Hg \rangle = \langle \nabla H^{-1}g, \nabla Hg \rangle, \\ \langle (H^{-1}g)', (Hg)' \rangle &= \langle H^{-1}g' - \lambda H^{-1}g, Hg' + \lambda Hg \rangle = \|g'\|_2^2 - \lambda^2\|g\|_2^2, \\ \langle \nabla H^{-1}g, \nabla Hg \rangle &= \|g'\|_2^2 + (k^2 - \lambda^2)\|g\|_2^2, \end{aligned}$$

2) Notice that $\overline{H^{-1}} = H, |H| = 1$,

$$\begin{aligned} -\langle \Delta g, \psi \rangle &= -\langle g, H^{-1}\Delta Hg \rangle = -\langle Hg, \Delta Hg \rangle = \langle \nabla Hg, \nabla Hg \rangle, \\ \langle (Hg)', (Hg)' \rangle &= \langle Hg' + i\lambda Hg, Hg' + i\lambda Hg \rangle = \|g'\|_2^2 + \lambda^2\|g\|_2^2 + \int_0^1 i\lambda(g\bar{g}' - g'\bar{g}), \end{aligned}$$

If $g(x) \in \mathbb{R}$, then the last term above is 0 because $g\bar{g}' = g'g$. So we have

$$\|g'\|_2^2 + (k^2 + \lambda^2)\|g\|_2^2 = \langle \nabla Hg, \nabla Hg \rangle = -\langle \Delta g, \psi \rangle = \langle \nabla g, \nabla \psi \rangle \leq \|\nabla g\|_2\|\nabla \psi\|_2,$$

and we get a slightly stronger inequality

$$\|\nabla \psi\|_2 \geq \frac{\|\nabla g\|_2^2 + \lambda^2\|g\|_2^2}{\|\nabla g\|_2} \geq \|\nabla g\|_2,$$

3) Assume $g(x) = u(x) + iv(x)$, the region that it encloses is Ω , then

$$\int_0^1 \frac{g\bar{g}' - g'\bar{g}}{2i} = \int_0^1 \Im(g\bar{g}') = \int_0^1 \Im(u + iv)(u' - iv') = \int_0^1 u'v - uv' = -2\text{Area}(\Omega),$$

where $\text{Area}(\Omega)$ denotes signed area with positive sign if $g(x)$ travels counter-clockwisely.

$$\int_0^1 i\lambda(g\bar{g}' - g'\bar{g}) = -2\lambda \int_0^1 \Im(g\bar{g}') = 4\lambda\text{Area}(\Omega),$$

i) By isoperimetric inequality on \mathbb{R}^2 , let $L(\partial\Omega) = \int_0^1 |g'|$ be the perimeter of $\partial\Omega$, we have

$$4\lambda Area(\Omega) \leq \frac{\lambda}{\pi} L(\partial\Omega)^2 = \frac{\lambda}{\pi} \|g'\|_1^2 \leq \frac{\lambda}{\pi} \|g'\|_2^2,$$

$$\langle \nabla Hg, \nabla Hg \rangle = \|\nabla g\|_2^2 + \lambda^2 \|g\|_2^2 + 4\lambda Area(\Omega) \geq \|\nabla g\|_2^2 + \lambda^2 \|g\|_2^2 - \frac{\lambda}{\pi} \|g'\|_2^2,$$

So when $0 < \lambda < \pi$, we have

$$\|\nabla g\|_2 \|\nabla \psi\|_2 \geq \langle \nabla g, \nabla \psi \rangle = \langle \nabla Hg, \nabla Hg \rangle \geq (1 - \frac{\lambda}{\pi}) \|\nabla g\|_2^2 + (\frac{k^2 \lambda}{\pi} + \lambda^2) \|g\|_2^2,$$

But the constant factor $C = 1 - \frac{\lambda}{\pi}$ on the right hand side depends on λ .

ii)

$$\int_0^1 \Im(g\bar{g}') \leq \int_0^1 |g||g'| \leq \|g\|_2 \|g'\|_2, \quad \langle \nabla Hg, \nabla Hg \rangle \geq \|\nabla g\|_2^2 + \lambda^2 \|g\|_2^2 - 2\lambda \|g\|_2 \|g'\|_2,$$

Directly use $\lambda^2 \|g\|_2^2 - 2\lambda \|g\|_2 \|g'\|_2 + \|g'\|_2^2 \geq 0$ exploits all first order term in $\|\nabla g\|_2^2$, but it is independent of λ . So we need a more delicate estimate.

iii) We want to show that for some $\epsilon > 0$,

$$\langle \nabla Hg, \nabla Hg \rangle = \|\nabla g\|_2^2 + \lambda^2 \|g\|_2^2 - 2\lambda \int_0^1 \Im(g\bar{g}') \geq \epsilon \|\nabla g\|_2^2,$$

$$\int_0^1 \Re(g\bar{g}') = \int_0^1 \frac{gg' + \bar{g}\bar{g}'}{2} = \int_0^1 (\frac{|g|^2}{2})' = 0,$$

So $\int_0^1 gg'$ is purely imaginary. Assume that $g = \sum_{j>0} \hat{g}_j e_j$, $e_j = \frac{\sqrt{2} \sin j\pi x}{j\pi}$, then

$$\begin{aligned} \int_0^1 gg' &= \int_0^1 (\sum_{j>0} \hat{g}_j \frac{\sqrt{2} \sin j\pi x}{j\pi}) (\sum_{l>0} \bar{\hat{g}}_l \sqrt{2} \cos l\pi x) = \int_0^1 \sum_{j,l>0} \hat{g}_j \bar{\hat{g}}_l \frac{\sin(j+l)\pi x + \sin(j-l)\pi x}{j\pi} \\ &= \sum_{j,l>0, 2\nmid j+l} \hat{g}_j \bar{\hat{g}}_l \frac{1}{j\pi} (\frac{2}{(j+l)\pi} + \frac{2}{(j-l)\pi}) = \sum_{j,l>0, 2\nmid j+l} \hat{g}_j \bar{\hat{g}}_l \frac{4}{(j^2 - l^2)\pi^2}, \end{aligned}$$

$$\sum_{j,l>0, 2\nmid j+l} \hat{g}_j \bar{\hat{g}}_l \frac{4}{(j^2 - l^2)\pi^2} = \sum_{j,l>0, 2\nmid j, 2\nmid l} \hat{g}_j \bar{\hat{g}}_l \frac{4}{(j^2 - l^2)\pi^2} - \sum_{j,l>0, 2\nmid j, 2\nmid l} \bar{\hat{g}}_j \hat{g}_l \frac{4}{(j^2 - l^2)\pi^2},$$

$$(1 - \epsilon) \|\nabla g\|_2^2 + \lambda \|g\|_2^2 = \sum_{j>0} |\hat{g}_j|^2 ((1 - \epsilon)(1 + \frac{k^2}{j^2 \pi^2}) + \frac{\lambda^2}{j^2 \pi^2}),$$

However, numerical simulation doesn't support that there exists $\epsilon > 0$ such that

$$(1 - \epsilon) \|\nabla g\|_2^2 + \lambda \|g\|_2^2 \geq 2\lambda \left| \int_0^1 gg' \right| = 2\lambda \left| \sum_{j,l>0, 2\nmid j+l} \hat{g}_j \bar{\hat{g}}_l \frac{4}{(j^2 - l^2)\pi^2} \right|,$$

holds uniformly for all $\lambda > 0$. □

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